DEVELOPMENTS

OBTAINED BY

CAUCHY'S THEOREM

WITH

APPLICATIONS TO THE ELLIPTIC FUNCTIONS

BY

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PREFACE.

The basis of this paper is an "*Extrait d'une lettre de M. Gomes Teixeira à M. Hermite*," published in the *Bulletin des Sciences Mathematiques* for Sept. 1890. He showed how, by the employment of Cauchy's theorem, we can get an analytical representation of a function in the form of a series going according to ascending powers of the sine. I have given his method in §§3–5, and my treatment of the question of Convergence is essentially the same as his, although the details have been somewhat simplified.

By reducing to another form the function whose residue gives the representation sought, I have been able to deduce the law of the series; but the forms which presented themselves in this process suggested another and much simpler form of function to start with, and by this I have obtained a general form of development in powers of any holomorphic function and one or two interesting theorems concerning these functions. These general formulae have been applied to get developments in powers of the tangent and of the sn, and I have given some of the different forms which these developments may take and some examples of their application. Finally, the formulae which give developments in powers of the sine and of the sn have been employed to obtain these developments for sn (mx), cn (mx), and dn (mx). In this application I have made use of the method of "ternary paths" due to M. Desiré André and employed by him in calculating the developments of $\mathrm{sn}^p x$, $\mathrm{cn}^p x$,

and $\mathrm{dn}^p x$ by Maclaurin's formula* (*Annales de L'Ecole Normale Supérieure*, 2 *Série*, T. 6).

Dr. Craig first called my attention to the communication of M. Teixeira and suggested to me the investigation which has led to these results, and I have had the benefit of his advice in all of this work.

BALTIMORE, *May* 1, 1891.

* The use of the function $\phi(x)$ to represent the three elliptic functions and the method of treating the cn (mx) given in §36 were also suggested by André's paper.

CONTENTS.

INTRODUCTION.

1. The possibility of the analytical representation of a function by means of a series, and the limits of convergence of such series, have been among the most difficult and important questions of analysis. And such representations are continually called for in the applications of mathematics, particularly to physical and astronomical problems. It was the question of a vibrating chord that gave rise to a form of development sought for by D'Alembert, Euler, Bernoulli, and Lagrange, but completely established first by Fourier in 1807.* There are many kinds of development more or less useful and interesting, but the difficulty of determining the limits of convergence and of applying a given form to a given case is often well-nigh insuperable. Many forms that have been given to the remainders, for example, of Taylor's and Maclaurin's formulae are difficult to test. It is only with the modern notion of the curvilinear integral and Cauchy's beautiful theorems that we have been able to arrive at a method of testing this matter which is easy and of wide application. Hermite† has shown how by means of the curvilinear integral we may obtain both of these formulae and at the same time simple criteria of their availability, and that, too, for the complex variable as well as the real variable. Finally, having these formulae and the notion of residue, we are able, by virtue of Cauchy's theorem, to obtain a great number of developments. This method has not been employed directly to obtain Fourier's series, but Dini, in his work already

* Ulisse Dini: *Serie di Fourier e Altre Representazioni Analitiche delle Funzione di Una Variabile Reale. Parte Prima*, §2.

† *Cour de M. Hermite, Quatrième édition*, 9e *Leçon.*

cited, in which he takes up the investigation of more general forms of representation that include Fourier's series as a special case, has made frequent use of this theorem, although applying his results only to the case of real variables.

In Dini's applications of Cauchy's theorem, and in many others that have given rise to various forms of development, particularly in the theory of the Elliptic Functions, the different terms of a series are obtained from the different points which are poles within the area of the contour of integration of some function suitably formed.

For example, for the function $f(x)$ given arbitrarily from 0 to $2K$ he finds the following representation (p. 283):

$$f(x) = \frac{k'}{\pi} \sum_{0}^{\infty}{}' \frac{\theta(x+\lambda_n)}{\theta(x)\theta^2(\lambda_n)(1-\zeta\,\mathrm{sn}^2\lambda_n)} \int_0^{2K} f(a)\frac{\theta(a-\lambda_n)}{\theta(a)}\,da,$$

where $\lambda_1\lambda_2 \ldots\ldots \lambda_n \ldots\ldots$ are the roots of $H'(z) = 0$ which are found on the axis of y and λ_0 the root K of the same equation; $\theta(z)$, $H(z)$, the well-known Jacobian functions and

$$\zeta = \frac{1}{K}\int_0^K k^2\,\mathrm{sn}^2 z\,dz.$$

2. In his application of Cauchy's theorem, Hermite considers the integral

$$\int_s \frac{f(z)(x-a)^m}{(z-x)(z-a)^m}\,dz,$$

and shows that it leads to Taylor's formula. He also makes use of the integra

$$\int_s \frac{f(z)\,F(x)\,dz}{(z-x)\,F(z)},$$

where $\qquad F(z) = (z-a)^{\alpha}(z-b)^{\beta} \ldots\ldots (z-l)^{\lambda},$

$\alpha, \beta, \ldots\ldots, \lambda$ being positive integers, and obtains from this the solution of a problem of interpolation. Teixeira's suggestion consists essentially in the substitution of the sine for the simple binomial, and similar series could be obtained by the use of other functions that have a suitable addition theorem. But a

simpler form, more analogous to that employed in obtaining Taylor's series, will give much more general results that include these.

We may add that M. Teixeira applies the same method to the integral

$$\int_s \frac{f(z)\sin(x-\alpha)\sin(x-\beta)\ldots\ldots\sin(x-\lambda)}{\sin(z-x)\sin(z-\alpha)\sin(z-\beta)\ldots\ldots\sin(z-\lambda)}\,dz,$$

and establishes conditions similar to those found for the sine series, for the convergence of the following formula due to Hermite:

$$\begin{aligned} f(x) &= \frac{\sin(x-\beta)\sin(x-\gamma)\ldots\ldots\sin(x-\lambda)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\ldots\ldots\sin(\alpha-\lambda)}f(\alpha) \\ &+ \frac{\sin(x-\alpha)\sin(x-\gamma)\ldots\ldots\sin(x-\lambda)}{\sin(\beta-\alpha)\sin(\beta-\gamma)\ldots\ldots\sin(\beta-\lambda)}f(\beta) \\ &+ \frac{\sin(x-\alpha)\sin(x-\beta)\ldots\ldots\sin(x-\lambda)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)\ldots\ldots\sin(\gamma-\lambda)}f(\gamma) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

PART I.

General Developments.

Developments in Powers of $\sin(x-a)$ obtained by means of a Curvilinear Integral.

3. Teixeira considers the curvilinear integral

$$J = \int_s \frac{f(z) \sin^m (x-a)\, dz}{\sin(z-x) \sin^m (z-a)},$$

taking for contour of integration the rectangle whose centre is the point which has for affix a, and whose sides are two right lines parallel to the axis of abscissae equal in length to π, and two right lines parallel to the axis of ordinates equal in length to $2l$; and he assumes that $f(z)$ is holomorphic in the area limited by this contour, and that x is the affix of a point in the interior of this contour.

Then the theorem of Cauchy gives an expression for this integral in the form

$$J = 2\pi i (A + B),$$

A and B being the residues of the function

$$F(z) = \frac{f(z) \sin^m (x-a)}{\sin(z-x) \sin^m (z-a)}$$

with respect to x and a, which are roots of

$$\sin(z-x) = 0$$

and

$$\sin(z-a) = 0,$$

in the interior of the area considered.

4. The residue of $F(z)$ with respect to x is the coefficient of $\frac{1}{h}$ in the development of

$$F(x+h) = \frac{f(x+h) \sin^m (x-a)}{\sin h \sin^m (x-a+h)}$$

in a series arranged according to ascending powers of h, and therefore we have

$$A = f(x).$$

The residue B of $F(z)$ with respect to a is the coefficient of $\frac{1}{h}$ in the development of

$$F(a+h) = \frac{f(a+h)\sin^m(x-a)}{\sin(a-x+h)\sin^m h}$$

in a series arranged according to ascending powers of h.

He shows that by developing the three functions

$$f(a+h), \quad \frac{1}{\sin(a-x+h)}, \quad \frac{h^m}{\sin^m h}$$

and multiplying the product of their developments by

$$\frac{\sin^m(x-a)}{h^m},$$

B is obtained in a form which enables us to deduce from our expression for f the following formulae:

$$\left\{\begin{aligned} f(x) = &\sum_{n=0}^{\frac{1}{2}m-1} K_{2n+1}\sin^{2n+1}(x-a) \\ &+ \cos(x-a)\sum_{n=0}^{\frac{1}{2}m-1} L_{2n}\sin^{2n}(x-a) \\ &+ \frac{1}{2\pi i}\int_s \frac{f(z)\sin^m(x-a)\,dz}{\sin(z-x)\sin^m(z-a)}, \end{aligned}\right.$$

when m is even; and

$$\left\{\begin{aligned} f(x) = &\sum_{n=0}^{\frac{1}{2}(m-1)} K'_{2n}\sin^{2n}(x-a) \\ &+ \cos(x-a)\sum_{n=0}^{\frac{1}{2}(m-3)} L'_{2n+1}\sin^{2n+1}(x-a) \\ &+ \frac{1}{2\pi i}\int_s \frac{f(z)\sin^m(x-a)\,dz}{\sin(z-x)\sin^m(z-a)}, \end{aligned}\right.$$

when m is odd.

5. Finally, the coefficients K and L as well as the fact that they are independent of m, may be obtained by putting $x = a$ in the above formulae and their successive derivatives with

respect to x. This gives for the coefficients of the first formula,

$$\begin{cases} L_0 = f(a), \\ K_1 = f'(a), \\ L_2 = \frac{1}{2}[f(a) + f''(a)], \\ K_2 = \frac{1}{6}[f'(a) + f'''(a)], \\ \dots\dots\dots\dots\dots \end{cases}$$

and for the second,

$$\begin{cases} K_0' = f(a), \\ L_1' = f'(a), \\ K_2' = \frac{1}{2}f''(a), \\ L_3' = \frac{1}{6}[f'''(a) + 4f'(a)], \\ \dots\dots\dots\dots\dots \end{cases}$$

The Question of Convergence.

6. The above formulae give two developments of $f(x)$ in series of ascending powers of $\sin(x-a)$ if the integral

$$J = \int_s \frac{f(z)\sin^m(x-a)\,dz}{\sin(z-x)\,\sin^m(z-a)},$$

tends towards zero when m tends towards infinity. But we have

$$J = \frac{\theta\sigma}{2\pi} \cdot \frac{f(\zeta)\sin^m(x-a)}{\sin(\zeta-x)\sin^m(\zeta-a)},$$

where ζ represents the affix of a point of the contour of integration, σ the perimeter of this contour, and θ a factor whose modulus is less than unity. Then if we have

$$|\sin(x-a)| < |\sin(z-a)|$$

on all points of the contour of integration, the integral J tends towards zero when m tends towards infinity, and the series are convergent.

7. If we put

$$z - a = \xi + i\eta,$$

and represent by M the modulus of $\sin(z-a)$, we have

$$\begin{aligned} M^2 &= \sin(\xi + i\eta)\sin(\xi - i\eta) \\ &= -\tfrac{1}{2}\cos 2\xi + \tfrac{1}{2}\cos 2i\eta \\ &= -\cos^2\xi + \cos^2 i\eta. \end{aligned}$$

Now let us seek the smallest value which M^2 can take when z describes the rectangle which constitutes this contour, that is, the contour given by the lines

$$\xi = \pm \frac{\pi}{2}, \text{ parallel to the axis of } y,$$

and $\eta = \pm l$, " " " x.

For both of the first two, M^2 reduces to

$$\cos^2 i\eta,$$

and its minimum value is 1 corresponding to $\eta = 0$, that is, to the middle points of these sides.

The second pair of sides also both give the same value to M^2; namely,

$$M^2 = \sin^2 \xi + (i \sin il)^2,$$

and its minimum value is

$$(i \sin il)^2 = \left(\frac{e^l - e^{-l}}{2}\right)^2,$$

which corresponds to $\xi = 0$, that is, to the middle points of these sides. It results that the minimum value which M^2 takes when z describes the contour of integration is the smaller of the quantities

$$1, \quad \left(\frac{e^l - e^{-l}}{2}\right)^2.$$

But $$\frac{e^l - e^{-l}}{2} > 1 \text{ if } l > \log(1 + \sqrt{2}),$$

and $$\frac{e^l - e^{-l}}{2} < 1 \text{ if } l < \log(1 + \sqrt{2}).$$

Thus we have the following theorem:

If $l \geqq \log(1 + \sqrt{2})$, the integral J tends towards zero when m tends towards infinity, if x satisfies the condition

$$|\sin(x - a)| < 1.$$

If $l < \log(1 + \sqrt{2})$, the integral J tends towards zero when m tends towards infinity, if x satisfies the condition

$$|\sin(x - a)| < \frac{e^l - e^{-l}}{2}.$$

In these two cases we can develop $f(x)$ in a convergent series by means of the formulae

$$f(x) = \sum_0^\infty K_{2n+1} \sin^{2n+1}(x-a) + \cos(x-a) \sum_0^\infty L_{2n} \sin^{2n}(x-a);$$

$$f(x) = \sum_0^\infty K'_{2n} \sin^{2n}(x-a) + \cos(x-a) \sum_0^\infty L'_{2n+1} \sin^{2n+1}(x-a).$$

The Nature of the Series and the Form of the Coefficients.

8. We have two representations for $f(x)$ which have the same area of convergence and therefore must be identical. Moreover, each consists of two parts, one involving only even powers and the other only odd powers of $x - a$. These parts, then, must be separately equal, and if we write

$$\begin{aligned} A &= \sum_0^\infty K_{2n+1} \sin^{2n+1}(x-a) &&= A_1, \text{ say} \\ &= \cos(x-a) \sum_0^\infty L_{2n+1} \sin^{2n+1}(x-a) &&= A_2, \end{aligned}$$

and
$$\begin{aligned} B &= \sum_0^\infty K_{2n} \sin^{2n}(x-a) &&= B_1 \\ &= \cos(x-a) \sum_0^\infty L_{2n} \sin^{2n}(x-a) &&= B_2, \end{aligned}$$

then
$$f(x) = A + B,$$

which may take any one of the four forms,

$$A_1 + B_2, \quad A_2 + B_1, \quad A_1 + B_1, \quad A_2 + B_2.$$

The last two give

$$f(x) = \sum_0^\infty K_n \sin^n(x-a)$$

and
$$f(x) = \cos(x-a) \sum_0^\infty L_n \sin^n(x-a).$$

The former of these two has the advantage of being the simplest of all the four forms; the latter is in integrable form, and, being a uniformly convergent series, can be integrated at once. We can verify that A_1 and A_2 are identical, also B_1 and B_2, by developing

$$\cos(x-a) = [1 - \sin^2(x-a)]^{\frac{1}{2}}$$

by the binomial formula and performing the multiplications indicated, when A_2 and B_2 will reduce at once to A_1 and B_1.

The values of A and B themselves may be obtained by combining $f(x)$ and $f(2a-x)$; in fact

$$2A = f(x) - f(2a-x),$$
$$2B = f(x) + f(2a-x).$$

A and B are both uniformly convergent, their remainders after m terms being the sum or difference of two integrals that diminish indefinitely as m tends towards infinity.

9. Instead of obtaining the coefficients K and L by the method indicated in §5, we may proceed in the following manner:

The factor $\dfrac{1}{\sin(z-x)}$ which occurs in the function $F(z)$ (§3), may be separated into four parts; namely, writing s, c, s', c' for $\sin(x-a)$, $\cos(x-a)$, $\sin(z-a)$, $\cos(z-a)$, we have

$$\frac{1}{\sin(z-x)} = \frac{1}{s'c - sc'} = \frac{s'c + sc'}{s'^2 - s^2}$$
$$= \frac{c}{2(s'-s)} + \frac{c}{2(s'+s)} + \frac{c'}{2(s'-s)} - \frac{c'}{2(s'+s)}.$$

Now each of the four integrals

$$\int_s \frac{f(z)\,cs^m dz}{(s'-s)\,s'^m},\quad \int_s \frac{f(z)\,cs^m dz}{(s'+s)\,s'^m},\quad \int_s \frac{f(z)\,c's^m dz}{(s'-s)\,s'^m},\quad \int_s \frac{f(z)\,c's^m dz}{(s'+s)\,s'^m},$$

taken over the same contour of integration as the integral considered in §3, becomes infinitely small when m increases beyond all limit, if we have

$$|\sin(x-a)| < |\sin(z-a)|$$

on all points of the contour of integration.

But we have the identities

$$\frac{s^m}{(s-s')\,s'^m} = \frac{1}{s'} + \frac{s}{s'^2} + \dots + \frac{s^{m-1}}{s'^m} + \frac{1}{s-s'}$$

and

$$-\frac{s^m}{(s+s')\,s'^m} = \frac{1}{s'} - \frac{s}{s'^2} + \dots \pm \frac{s^{m-1}}{s'^m} - \frac{1}{s+s'}.$$

Within the contour of integration $\frac{1}{s-s'}$ becomes infinite for $z = x$ and its residue is $-\frac{1}{c}$; $\frac{1}{s+s'}$ becomes infinite for $z = 2a - x$, and its residue is $\frac{1}{c}$. All the remaining terms become infinite only for $z = a$. Therefore we have

$$f(x) = c\sum_0^\infty L_n s^n \qquad = B_2 + A_2,$$

$$f(2a - x) = c\sum_0^\infty (-1)^n L_n s^n = B_2 - A_2,$$

$$f(x) = \sum_0^\infty K_n s^n \qquad = B_1 + A_1,$$

$$f(2a - x) = \sum_0^\infty (-1)^n K_n s^n = B_1 - A_1;$$

and we see readily that $A_1 = A_2$ and $B_1 = B_2$.

Here

$$L_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\sin^{n+1}(z-a)},$$

and

$$K_n = \frac{1}{2\pi i}\int_s \frac{f(z)\cos(z-a)\,dz}{\sin^{n+1}(z-a)} = \frac{D}{n} L_{n-1}$$

where D stands for $\frac{d}{da}$. [Thoughout this paper we shall use D to denote differentiation with respect to a.]

If we differentiate K_{n-1} with respect to a, we shall get

$$\frac{D^2}{n-1} L_{n-2} = \frac{1}{2\pi i}\int_s \frac{f(z)[\sin^2(z-a) + n\cos^2(z-a)]\,dz}{\sin^{n+1}(z-a)}$$

$$= -(n-1)L_{n-2} + nL_n,$$

$$\therefore\ L_n = \frac{D^2 + (n-1)^2}{n(n-1)} L_{n-2}.$$

Now evidently

$$L_0 = f(a),\quad L_1 = f'(a),\quad K_0 = f(a);$$

hence in general

$$L_n = \frac{1}{n!}[D^2 + (n-1)^2][D^2 + (n-3)^2]\ldots\ldots,$$

the last factors being

$$(D^2 + 1^2)f(a) \text{ when } n \text{ is even,}$$

and

$$(D^2 + 2^2)\, D f(a) \text{ `` `` odd;}$$

$$K_n = \frac{1}{n!}\, [D^2 + (n-2)^2][D^2 + (n-4)^2] \ldots .,$$

the last factors being

$$(D^2 + 2^2)\, D^2 f(a) \text{ when } n \text{ is even,}$$

and

$$(D^2 + 1^2)\, D f(a) \text{ `` `` odd.}$$

A involves only odd powers of $\sin(x - a)$ and odd derivatives of $f(a)$, and B only even powers of $\sin(x - a)$ and even derivatives of $f(a)$.

Applications.

10. M. Teixeira has taken for examples $\sin kx$ and $\cos kx$, and by employing the formulae given in §5, deduced the following well-known developments:

$$\sin kx = k \sin x - \frac{k(k^2 - 1^2)}{3!} \sin^3 x + \frac{k(k^2 - 1^2)(k^2 - 3^2)}{5!} \sin^5 x - \ldots .,$$

$$\sin kx = \cos x \left[k \sin x - \frac{k(k^2 - 2^2)}{3!} \sin^3 x + \ldots . \right],$$

$$\cos kx = 1 - \frac{k^2}{2} \sin^2 x + \frac{k^2(k^2 - 2^2)}{4!} \sin^4 x - \ldots .,$$

$$\cos kx = \cos x \left[1 - \frac{k^2 - 1^2}{2} \sin^2 x + \frac{(k^2 - 1^2)(k^2 - 3^2)}{4!} \sin^4 x - \ldots . \right].$$

In fact the even derivatives of $\sin x$ and the odd derivatives of $\cos x$ vanish with x, while the derivatives which do not vanish become simply the corresponding powers of k with alternating signs; hence we have only to replace D by k in the formulae given in §9 and properly determine the signs, and we get these developments at once.

Similarly with exponentials, we have but to replace D by k in all the formulae for L_n and K_n and we get

$$e^{kx} = A + B,$$

where A may take either of the forms

$$A_1 = k \sin x + \frac{k(k^2 + 1^2)}{3!} \sin^3 x + \frac{k(k^2 + 1^2)(k^2 + 3^2)}{5!} \sin^5 x + \ldots\ldots,$$

$$A_2 = \cos x \left[k \sin x + \frac{k(k^2 + 2^2)}{3!} \sin^3 x + \ldots\ldots \right];$$

and B either of these two,

$$B_1 = 1 + \frac{k^2}{2} \sin^2 x + \frac{k^2(k^2 + 2^2)}{4!} \sin^4 x + \ldots\ldots,$$

$$B_2 = \cos x \left[1 + \frac{k^2 + 1^2}{2} \sin^2 x + \frac{(k^2 + 1^2)(k^2 + 3^2)}{4!} \sin^4 x + \ldots\ldots \right].$$

Notice that A_2 and B_2 may be obtained from B_1 and A_1 by differentiation, and so also may the second formulae for $\sin kx$ and $\cos kx$ be obtained from the first formulae for $\cos kx$ and $\sin kx$.

These series are all convergent as long as

$$|\sin x| < 1$$

and the real part of x lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. k may be anything whatever.

11. Let us consider the function $f(x) = x$; then since $f(x)$ vanishes with x and all the derivatives are zero except the first, which is unity, we get at once the two developments A_1 and A_2, namely,

$$x = \sin x + \frac{1^2}{3!} \sin^3 x + \frac{1^2 . 3^2}{5!} \sin^5 x + \ldots\ldots,$$

$$x = \cos x \left[\sin x + \frac{2^2}{3!} \sin^3 x + \frac{2^2 . 4^2}{5!} \sin^5 x + \ldots\ldots \right].$$

In the same way we may obtain the following:

$$x^2 = 2\cos x\left[\frac{1}{2}\sin^2 x + \frac{1^2+2^2}{4!}\sin^4 x + \frac{1^2.3^2+1^2.5^2+3^2.5^2}{6!}\sin^6 x + \dots\right].$$

$$x^2 = 2\left[\frac{1}{2}\sin^2 x + \frac{2^2}{4!}\sin^4 x + \frac{2^2.4^2}{6!}\sin^6 x + \dots\right].$$

If we put $f(x) = \text{const.}$, we shall find that constant a factor in our development, and removing it, we get the identity

$$1 = \cos x\left[1 + \frac{1}{2}\sin^2 x + \frac{1^2.3^2}{4!}\sin^4 x + \dots\right],$$

which agrees with the result of developing

$$\frac{1}{\cos x} = (1 - \sin^2 x)^{-\frac{1}{2}}$$

by the binomial formula.

All these formulae hold as long as

$$|\sin x| < 1$$

and the real part of x lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Developments in Powers of $\cos(x - a)$.

12. Put $a = b + \frac{\pi}{2}$, then

$$\sin(x-a) = -\cos(x-b),$$
$$\cos(x-a) = \sin(x-b),$$

and we can substitute these expressions and get series going according to the powers of $\cos(x-b)$, with the same areas of convergence as the above. Or we may move the areas of convergence to the right a distance equal to $\frac{\pi}{2}$ and write

$$A_1 = -\sum_0^\infty K_{2n+1} \cos^{2n+1}(x-a),$$
$$A_2 = -\sin(x-a) \sum_0^\infty L_{2n+1} \cos^{2n+1}(x-a),$$
$$B_1 = \sum_0^\infty K_{2n} \cos^{2n}(x-a),$$
$$B_2 = \sin(x-a) \sum_0^\infty L_{2n} \cos^{2n}(x-a),$$

where L_n and K_n have the same form as in §9 except that $f\left(a+\frac{\pi}{2}\right)$ is to be substituted for $f(a)$.

In particular for $\sin kx$ and $\cos kx$,

$$A_1 = -\left[k\cos x - \frac{k(k^2-1^2)}{3!}\cos^3 x + \frac{k(k^2-1^2)(k^2-3^2)}{5!}\cos^5 x - \ldots\ldots\right],$$
$$A_2 = -\sin x\left[k\cos x - \frac{k(k^2-2^2)}{3!}\cos^3 x + \ldots\ldots\right],$$
$$B_1 = 1 - \frac{k^2}{2}\cos^2 x + \frac{k^2(k^2-2^2)}{3!}\cos^4 x - \ldots\ldots,$$
$$B_2 = \sin x\left[1 - \frac{k^2-1^2}{2}\cos^2 x + \frac{(k^2-1^2)(k^2-3^2)}{4!}\cos^4 x - \ldots\ldots\right];$$

and

$$\sin kx = \cos\frac{k\pi}{2}.A + \sin\frac{k\pi}{2}.B,$$
$$\cos kx = -\sin\frac{k\pi}{2}.A + \cos\frac{k\pi}{2}.B;$$

where we may take either expression of A with either expression of B, and where k may be anything whatever. These formulae hold as long as

$$|\cos x| < 1$$

and the real part of x lies between 0 and π.

We might give other applications like those in §11.

More General Formulae.

13. We can get our results in another way without the use of an addition theorem, and the method being applicable to a much larger number of functions, we will take a more general point of view.

Consider the integral

$$\frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\varphi(z)-\varphi(x)},$$

where we assume that within the contour of integration $f(z)$ and $\varphi(z)$ are holomorphic, $\varphi(z)$ vanishes for $z=a$, and the equation $\varphi(z)=\varphi(x)$ has the one root $z=x$. The value of this integral will evidently be $\dfrac{f(x)}{\varphi'(x)}$. But from the identity

$$\frac{1}{\varphi(z)-\varphi(x)} = \frac{1}{\varphi(z)} + \frac{\varphi(x)}{\varphi^2(z)} + \dots + \frac{\varphi^{m-1}(x)}{\varphi^m(z)} + \frac{\varphi^m(x)}{[\varphi(z)-\varphi(x)]\,\varphi^m(z)}$$

we get the development

$$f(x) = \varphi'(x)\left[\sum_0^{m-1} J_n\varphi^n(x) + R_m\right],$$

where

$$J_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\varphi^{n+1}(z)}$$

and

$$R_m = \frac{1}{2\pi i}\int_s \frac{f(z)\,\varphi^m(x)\,dz}{[\varphi(z)-\varphi(x)]\,\varphi^m(z)} = \frac{\theta\sigma}{2\pi}\,\frac{f(\zeta)}{[\varphi(\zeta)-\varphi(x)]}\left(\frac{\varphi(x)}{\varphi(\zeta)}\right)^m,$$

θ, σ and ζ having the same significance as in §6.

Now if $$|\varphi(x)| < |\varphi(z)|$$

for all points on the contour of integration, this remainder vanishes when m becomes infinite.

14. Instead of $\varphi(z)$ we might have taken the form $\varphi(z-a)$ suggesting the root a. We should have obtained the formula

$$f(x) = \varphi'(x-a)\sum_0^{\infty} L_n\varphi^n(x-a)$$

where
$$L_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\varphi^{n+1}(z-a)}.$$

We might also have started with the integral

$$\frac{1}{2\pi i}\int_s \frac{f(z)\,\varphi'(z-a)\,dz}{\varphi(z-a)-\varphi(x-a)}$$

under the same conditions, and obtained the series

$$f(x) = \sum_0^\infty K_n\varphi^n(x-a)$$

where
$$K_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,\varphi'(z-a)\,dz}{\varphi^{n+1}(z-a)} = \frac{D}{n}L_{n-1}.$$

Evidently also, if we suppose a is a simple root of $\varphi(z-a)$ and the only root within the contour,

$$L_0 = \frac{f(a)}{\varphi'(0)},\quad K_0 = f(a),$$

and
$$L_1 = \frac{f'(a)\,\varphi'(0) - f(a)\,\varphi''(0)}{\varphi'^3(0)}.$$

If we differentiate K_{n-1} with respect to a we shall get

$$\frac{D^2}{n-1}L_{n-2} = \frac{1}{2\pi i}\int_s \frac{f(z)[n\varphi'^2(z-a) - \varphi(z-a)\varphi''(z-a)]\,dz}{\varphi^{n+1}(z-a)}.$$

Now if φ satisfies the relation

$$\varphi'^2 = \Phi(\varphi),\ \text{a polynomial in } \varphi,$$

and
$$\therefore\ 2\varphi'' = \Phi'(\varphi),\ \text{also a polynomial in } \varphi,$$

this formula will give us a relation among the coefficients of the development by which they can be all obtained from the first one, two or more of them. This applies not only to the sine, tangent and sn, but also to the hyperelliptic functions. It is not necessary to use this in the case of the tangent, for in the expression of K_n the value of φ' may be written at once and we shall get a relation connecting three successive coefficients of the tangent development.

Instead of $\varphi(z) - \varphi(x)$ in the denominator of the expression with which we started, we might have taken any other form with two terms which has a root given in terms of x and existing within the contour of integration. For instance, if we use the form $\varphi(z-a)$ and make the contour symmetrical about the point a, we may take for our denominator a function which has $2a - x$ for a root. In case $\varphi(z-a)$ is an odd function, this will be

$$\varphi(z-a) + \varphi(x-a)$$

and will give us developments for $f(2a - x)$ exactly as we have obtained them in the case of the sine. Of course these developments will differ from the corresponding series for $f(x)$ only in the signs of the alternate terms, and we can show that all of our series can be separated into two parts, each of which is uniformly convergent, and which may be combined in four different ways precisely as in §8.

Developments in Powers of tan $(x - a)$.

15. We will consider the case where our φ is the function $\tan(z-a)$ and put

$$z - a = \xi + i\eta.$$

Take for contour of integration the rectangle formed by the lines

$$\xi = \pm \frac{\pi}{4} \text{ parallel to the axis of } y,$$

and $\qquad \eta = \pm l \qquad$ " " " x.

The modulus M of $\tan(z-a)$ will be given by

$$M^2 = \tan(\xi + i\eta)\tan(\xi - i\eta)$$
$$= \frac{\tan^2\xi - \tan^2 i\eta}{1 - \tan^2\xi \tan^2 i\eta}.$$

For $\xi = \pm \frac{\pi}{4}$ this reduces to

$$\frac{1 - \tan^2 i\eta}{1 - \tan^2 i\eta} = 1,$$

which is independent of the value of η.

For $\eta = \pm l$, M^2 reduces to

$$\frac{\tan^2 \xi + \left(\dfrac{e^l - e^{-l}}{e^l + e^{-l}}\right)^2}{1 + \tan^2 \xi \left(\dfrac{e^l - e^{-l}}{e^l + e^{-l}}\right)^2}$$

$$= \frac{e^{2l} + e^{-2l} - 2\cos 2\xi}{e^{2l} + e^{-2l} + 2\cos 2\xi},$$

which is a minimum when $\xi = 0$, since for ξ between $-\dfrac{\pi}{4}$ and $+\dfrac{\pi}{4}$ $\cos 2\xi$ is always positive. M^2 then reduces to

$$\left(\frac{e^l - e^{-l}}{e^l + e^{-l}}\right)^2,$$

which is always less than 1.

Therefore our results hold if x satisfies the condition

$$\tan (x - a) < \frac{e^l - e^{-l}}{e^l + e^{-l}},$$

and we may write

$$f(x) = \sum_0^\infty K_n \tan^n (x - a),$$

and $$f(x) = \sec^2 (x - a) \sum_0^\infty L_n \tan^n (x - a).$$

Now

$$K_n = \frac{D}{n} L_{n-1} = \frac{1}{2\pi i} \int_s \frac{f(z)[1 + \tan^2 (z - a)]\, dz}{\tan^{n+1} (z - a)}$$

$$= L_n + L_{n-2},$$

i. e. $$L_n = \frac{D}{n} L_{n-1} - L_{n-2};$$

and, changing n into $n - 1$ and operating with $\dfrac{D}{n}$, we get

$$K_n = \frac{D}{n} K_{n-1} - \frac{n-2}{n} K_{n-2}.$$

Also $$L_0 = K_0 = f(a),$$

$$L_1 = K_1 = f'(a).$$

We could put these expressions for L_n and K_n in another form with only even suffixes or odd suffixes in any one formula. For we have

$$\frac{D}{n} L_{n-1} = \frac{D^2}{n(n-1)} L_{n-2} - \frac{n-2}{n}(L_{n-2} + L_{n-4})$$
$$= \frac{D^2 - (n-1)(n-2)}{n(n-1)} L_{n-2} - \frac{n-2}{n} L_{n-4};$$
$$\therefore\ L_n = \frac{D^2 - 2(n-1)^2}{n(n-1)} L_{n-2} - \frac{n-2}{2} L_{n-4},$$

and
$$K_n = \frac{D^2 - 2(n-2)^2}{n(n-1)} K_{n-2} - \frac{(n-3)(n-4)}{n(n-1)} K_{n-4}.$$

Each of the tangent series, like the sine series, can be separated into two parts, one involving only odd powers of $\tan(x-a)$ and odd derivatives of $f(a)$, and the other only even powers of $\tan(x-a)$ and even derivatives of $f(a)$, and these may be combined in four different ways.

Applying these formulae to $\sin kx$ and $\cos kx$, we have the following known forms:

$$\sin kx = k\tan x - \frac{k(k^2-2)}{3!}\tan^3 x + \frac{k(k^4-20k^2+24)}{5!}\tan^5 x - \ldots,$$

$$\sin kx = \sec^2 x\left[k\tan x - \frac{k(k^2-8)}{3!}\tan^3 x + \ldots\right],$$

$$\cos kx = 1 - \frac{k^2}{2}\tan^2 x + \frac{k^2(k^2-8)}{4!}\tan^4 x - \ldots,$$

$$\cos kx = \sec^2 x\left[1 - \frac{k^2-2}{2}\tan^2 x + \frac{k^4-20k^2+24}{4!}\tan^4 x - \ldots\right],$$

which hold for any value whatever of k as long as the real part of x lies between $-\frac{\pi}{4}$ and $+\frac{\pi}{4}$.

16. One advantage of a series in powers of $\tan(x-a)$ is that it can be reduced at once to a similar series multiplied by $\sec^2(x-a)$ and so to integrable form.

Thus if we have a series of the form

$$A_0 + A_1 \tan x + A_2 \tan^2 x + \dots\dots,$$

and divide by $1 + \tan^2 x$, we get without difficulty the equivalent form

$$\sec^2 x\,[B_0 + B_1 \tan x + B_2 \tan^2 x + \dots\dots],$$

where the B's are given by the relation

$$B_n = A_n - B_{n-2},$$

with $$B_0 = A_0, \quad B_1 = A_1.$$

Of course it would be necessary after integration to consider the limits of convergence, but, in general, series obtained by integration are more rapidly convergent than those from which they were obtained.

Suppose we integrate the series

$$f(x) = \sec^2(x-a)[L_0 + L_1 \tan(x-a) + L_2 \tan^2(x-a) + \dots].$$

We obtain the indefinite integral

$$\int f(x)\,dx = C + L_0 \tan(x-a) + \tfrac{1}{2} L_1 \tan^2(x-a) + \dots\dots$$
$$= \sec^2(x-a)[M_0 + M_1 \tan(x-a) + M_2 \tan^2(x-a) + \dots\dots],$$

where

$$M_n = \frac{1}{n} L_{n-1} - M_{n-2}.$$

Therefore

$$DM_n = L_n + L_{n-2} - DM_{n-2},$$

or $$DM_n - L_n = -(DM_{n-2} - L_{n-2})$$
$$= \dots\dots = \pm (DM_1 - L_1),$$

or $$= \pm (DM_0 - L_0).$$

Now $M_0 = C$ and C is the value which the indefinite integral $\int f(x)\,dx$ takes when we put in it $x = a$,

$$\therefore\ DM_0 = DC = f(a)^* = L_0.$$

* If we integrate from some other limit than a, say the integral is $\int_b^x f(x)\,dx$, then $C = \int_b^a f(x)\,dx$, and we still have $DC = f(a)$.

Also $$M_1 = L_0 = f(a)$$
and $$\therefore\ DM_1 = f'(a) = L_1.$$
Therefore, for all values of n,
$$DM_n = L_n.$$
But
$$M_n = \frac{1}{n} L_{n-1} - M_{n-2},$$
$$\therefore\ M_n = \frac{D}{n} M_{n-1} - M_{n-2}.$$
This is the same relation that held among the L's, and we can repeat the process and get
$$\iint f(x)\,dx^2 = \sec^2(x-a)[N_0 + N_1 \tan(x-a) + \ldots.].$$
The first N's are readily determined and in general
$$N_n = \frac{D}{n} N_{n-1} - N_{n-2}.$$

This process may be repeated as long as the series remains convergent.

On the other hand, if we wish to get an expression for $\int f(x)\,dx$, we may treat this integral as our function, since there is no coefficient in the development
$$\sum_0^\infty K_n \tan^n(x-a)$$
except the first which does not involve the process of differentiation. The same is true of the more general form
$$\sum_0^\infty K_n \varphi^n(x-a)$$
given in §14.

We notice that this formula agrees with the first given above for $\int f(x)\,dx$, if we make
$$nK_n = L_{n-1},$$
and in this form the relation connecting the K's assumes the

same form as that connecting the L's, M's and N's given above, namely,

$$[nK_n] = \frac{D}{n-1}[(n-1)K_{n-1}] - [(n-2)K_{n-2}].$$

Developments in Powers of sn $(x-a)$.

17. Again, let our function φ be the function $\operatorname{sn}(z-a)$. Putting

$$z - a = K\xi + iK'\eta,$$

where ξ and η are always real, we will take for contour of integration the parallelogram whose sides are

the two lines $\xi = \pm 1$,

and " " " $\eta = \pm l$,

where l is positive and less than unity.

When k is real and $k^2 < 1$, the modulus of $\operatorname{sn}(z-a)$, say M, is given by

$$M^2 = \operatorname{sn}(K\xi + iK'\eta)\operatorname{sn}(K\xi - iK'\eta)$$

$$= \frac{\operatorname{sn}^2(K\xi) - \operatorname{sn}^2(iK'\eta)}{1 - k^2\operatorname{sn}^2(K\xi)\operatorname{sn}^2(iK'\eta)}.$$

This for the first two sides, $\xi = \pm 1$, reduces to

$$\frac{\operatorname{cn}^2(iK'\eta)}{\operatorname{dn}^2(iK'\eta)} = \frac{1}{dn^2(K'\eta, k')},$$

which varies, always increasing from 1 for $\eta = 0$ to $\frac{1}{k^2}$ for $\eta = 1$. Therefore the smallest value of M along these two sides is 1, which corresponds to their middle points.

For the other pair of sides $\eta = \pm l$, and we may put our expression for M^2 into the form

$$M^2 = \frac{\operatorname{sn}^2(K\xi) + [i\operatorname{sn}(iK'l)]^2}{1 + k^2\operatorname{sn}^2(K\xi).[i\operatorname{sn}(iK'l)]^2}.$$

$\operatorname{sn}^2 K\xi$ increases from 0 for $\xi = 0$ to 1 for $\xi = 1$. Consider the expression

$$y = \frac{x^2 + a^2}{1 + k^2a^2x^2},$$

where x and a are real and x^2 varies from 0 to 1. Its derivative

$$\frac{dy}{dx} = \frac{2x(1 - k^2a^4)}{(1 + k^2a^2x^2)^2}$$

is positive when x is positive and $a^2 < \frac{1}{k}$, and in that case the minimum value of y will be a^2 corresponding to $x = 0$.

We see then that M^2 will never be less than 1, the minimum already found on the first pair of sides, unless

$$[i \operatorname{sn} iK'l]^2 < 1,$$

i. e. unless

$$l < l_1,$$

where

$$[i \operatorname{sn} iK'l_1]^2 = \frac{\operatorname{sn}^2(K'l_1, k')}{\operatorname{cn}^2(K'l_1, k')} = 1,$$

whence

$$\operatorname{sn}^2(K'l_1, k') = \tfrac{1}{2},$$

or

$$l_1 = \frac{1}{K'}\int_0^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{1 - u^2 . 1 - k'^2u^2}} = \frac{\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{du}{\sqrt{1 - u^2 . 1 - k'^2u^2}}}{\displaystyle\int_0^1 \frac{du}{\sqrt{1 - u^2 . 1 - k'^2u^2}}}.$$

Therefore, when k is real and $k^2 < 1$, our results will hold as long as

$$|\operatorname{sn}(x - a)| < 1 \text{ when } l \overline{>} l_1,$$

and

$$|\operatorname{sn}(x - a)| < (i \operatorname{sn} iK'l)^2 \text{ when } l < l_1,$$

if also the real part of $x - a$ lies between $-K$ and $+K$.

In the most general case, when k is any complex quantity, it is impossible to fix any definite expression for the limit of convergence. We can only say that

$$|\operatorname{sn}(x - a)|$$

must be less than the smallest value that

$$|\operatorname{sn}(K\xi + iK'\eta)|$$

can have either when ξ varies from 0 to 1, and η takes some arbitrary value l between 0 and 1, or when $\xi = 1$ and η varies

from 0 to l; and, further, $x-a$ being written in the same form (viz. $K\xi + iK'\eta$), that the coefficient of K must lie between -1 and $+1$.

18. Therefore, when the conditions just defined are satisfied, we may write

$$f(x) = \sum_0^\infty K_n \operatorname{sn}^n(x-a)$$

and
$$f(x) = \operatorname{cn}(x-a)\operatorname{dn}(x-a)\sum_0^\infty L_n \operatorname{sn}^n(x-a),$$

where
$$K_n = \frac{D}{n} L_{n-1},$$

and L_n is obtained from the formula

$$\frac{D^2}{n-1} L_{n-2} = \frac{1}{2\pi i}\int_c \frac{f(z)[n \operatorname{cn}^2(z-a)\operatorname{dn}^2(z-a) + \operatorname{sn}^2(z-a)\{\operatorname{dn}^2(z-a)+k^2\operatorname{cn}^2(z-a)\}]dz}{\operatorname{sn}^{n+1}(z-a)}$$
$$= nL_n - (n-1)(1+k^2)L_{n-2} + (n-2)k^2L_{n-4}.$$

Therefore
$$L_n = \frac{D^2 + (n-1)^2(1+k^2)}{n(n-1)} L_{n-2} - \frac{(n-2)k^2}{n} L_{n-4}.$$

Then, changing n into $n-1$ and operating with $\frac{D}{n}$, we get

$$K_n = \frac{D^2 + (n-2)^2(1+k^2)}{n(n-1)} K_{n-2} - \frac{(n-3)(n-4)k^2}{n(n-1)} K_{n-4}.$$

Also
$$L_0 = f(a), \quad L_1 = f'(a), \quad K_0 = f(a),$$
$$L_2 = \frac{D^2 + (1+k^2)}{2} f(a), \quad L_3 = \frac{[D^2 + 2^2(1+k^2)]D}{3!} f(a).$$

Results obtained from the General Formulae.

19. We have given two formulae for the development of $f(x)$ in powers of $\varphi(x-a)$, namely,

$$f(x) = \sum_0^\infty K_n \varphi^n(x-a)$$

and
$$f(x) = \varphi'(x-a)\sum_0^\infty L_n \varphi^n(x-a),$$

where
$$K_n = \frac{D}{n} L_{n-1},$$

and
$$L_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\varphi^{n+1}(z-a)}.$$

Now
$$\frac{f(a+h)}{\varphi^{n+1}(h)} = \frac{1}{h^{n+1}}\left[\left(\frac{h}{\varphi(h)}\right)^{n+1} f(a+h)\right],$$

$$\therefore\ L_n = \frac{1}{n!}\left[\frac{d^n}{dh^n}\left\{\left(\frac{h}{\varphi(h)}\right)^{n+1} f(a+h)\right\}\right]_{h=0}$$
$$= \Sigma\frac{f^{(u)}(a)}{u!\,v!}\left[\frac{d^v}{dh^v}\left(\frac{h}{\varphi(h)}\right)^{n+1}\right]_{h=0},$$

the summation extending over all positive integer solutions (including zero) of the equation

$$u + v = n.$$

Also
$$K_n = \frac{1}{n!}\left[\frac{d^{n-1}}{dh^{n-1}}\left\{\left(\frac{h}{\varphi(h)}\right)^n f'(a+h)\right\}\right]_{h=0}$$
$$= \Sigma\frac{f^{(u+1)}(a)}{nu!\,v!}\left[\frac{d^v}{dh^v}\left(\frac{h}{\varphi(h)}\right)^n\right]_{h=0}$$

with
$$u + v = n - 1.$$

This last formula may be regarded as a transformation of Maclaurin's series. The formula is given by Schlömilch,* but he obtained it from the formulae for the differentiation of implicit functions and the change of the independent variable, by reductions which are pretty complicated.

If $z = f(x)$ and $y = \varphi(x)$, we can write an expression for $\dfrac{d^n z}{dy^n}$.

Suppose we put
$$Y = \varphi(x) - \varphi(a) = y - \varphi(a),$$

we have
$$\left(\frac{d^n f(x)}{dY^n}\right)_{Y=0} = \left[\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x-a}{\varphi(x)-\varphi(a)}\right)^n f'(x)\right]_{x=a}.$$

*Vorlesungen über einzelne Theile der Höhern Analysis, p. 22.

But

$$\left(\frac{d^n f(x)}{dY^n}\right)_{Y=0} = \left(\frac{d^n f(x)}{dy^n}\right)_{y=\phi(a)} = \frac{d^n f(a)}{d\{\varphi(a)\}^n}.$$

Hence, finally, changing x into ξ and a into x, we may write

$$\frac{d^n z}{dy^n} = \left[\frac{d^{n-1}}{d\xi^{n-1}}\left(\frac{\xi - x}{\varphi(\xi) - \varphi(x)}\right)^n f'(\xi)\right]_{\xi=x}.$$

This formula and others given by Schlömilch are readily obtained from our general formulae.*

By writing $f(x) = x$ we can use the series

$$x = \sum_0^\infty K_n \varphi^n (x - a)$$

to get the inverse of $\varphi(x - a)$, say Φ, by Maclaurin's formula. (Thus the first development in §11 gives us $\sin^{-1} y$ in powers of y, if we make $y = \sin x$.) In this case $K_0 = a$ and the inverse is

$$\Phi(x) = \sum_1^\infty K_n x^n,$$

where

$$K_n = \frac{1}{n!}\left[\frac{d^{n-1}}{dh^{n-1}}\left(\frac{h}{\varphi(h)}\right)^n\right]_0.$$

This might be used in getting the inversion of integrals, if the tests for convergence can be applied.

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* Another of these formulae given by Schlömilch may be written thus:

$$\frac{d^n z}{dx^n} = \sum_{s=1}^{s=n-1} \frac{A_{n,s}}{s!} \frac{d^s z}{dy^s}$$

where $A_{n,s} =$ limit when $\rho = 0$ of $\frac{d^n}{d\rho^n} \{\phi(x+\rho) - \phi(x)\}^s$. This is employed in obtaining Forsyth's canonical form for linear differential equations (see Craig's *Linear Differential Equations*, Vol. I, p. 464. It was from this that my attention was first called to the formulae of Schlömilch). By giving different values to n we get a series of equations linear in the derivatives of z with respect to y, and it is *by their solution* that Schlömilch obtained the relations expressed by the formula in the text.

20. Drop a for a moment and revert to the formula (§13),

$$f(x) = \varphi'(x)\left[\sum_0^{m-1} J_n \varphi^n(x) + R_m\right]$$

where R_m vanishes when m becomes infinitely large and

$$J_n = \frac{1}{2\pi i}\int_s \frac{f(z)\,dz}{\varphi^{n+1}(z)}.$$

This can be obtained without supposing that $\varphi(z)$ has a root within the contour of integration. But if $\varphi(z)$ had no such root, all the J's would be zero and $f(x)$ would vanish with R_m, which is absurd, since $f(x)$ is entirely independent of the form of φ. Furthermore, the method by which this formula was obtained will apply equally well if $\varphi(z)$ has one or more poles within the contour, provided x is kept away from those points. Therefore we have the following theorem:

If $\varphi(z)$ is a function uniform and having no singular essential point within a given contour, and if there exists a point within this contour for which the modulus of φ is less than for all points on the contour itself, then $\varphi(z)$ possesses a root within the contour.

Conversely, it is obvious that we can find a contour embracing a root of a uniform function such that for points sufficiently near this root the modulus of the function is less than for all points on the contour, since the roots of a uniform function are isolated.

Hence any function which is holomorphic in the region of a point may be developed in a series going according to ascending powers of any other holomorphic function which vanishes at that point.

PART II.

DEVELOPMENTS OF sn (mx), cn (mx), dn (mx).

The General Form of these Developments.

21. Taking the point a at the origin, we have found the following forms of development:

$$A_1 = \sum_0^\infty{}' K_{2n+1} \sin^{2n+1} x, \qquad C_1 = \sum_0^\infty{}' K_{2n+1} \operatorname{sn}^{2n+1} x,$$

$$A_2 = \cos x \sum_0^\infty{}' L_{2n+1} \sin^{2n+1} x, \quad C_2 = \operatorname{cn} x \operatorname{dn} x \sum_0^\infty{}' L_{2n+1} \operatorname{sn}^{2n+1} x,$$

$$B_1 = \sum_0^\infty{}' K_{2n} \sin^{2n} x, \qquad D_1 = \sum_0^\infty{}' K_{2n} \operatorname{sn}^{2n} x,$$

$$B_2 = \cos x \sum_0^\infty{}' L_{2n} \sin^{2n} x; \qquad D_2 = \operatorname{cn} x \operatorname{dn} x \sum_0^\infty{}' L_{2n} \operatorname{sn}^{2n} x,$$

and for x within a certain area of convergence we have

$$f(x) = A + B$$

and

$$f(x) = C + D,$$

where with each letter we may use either subscript 1 or 2.

Since A and C involve only odd powers of x and in their coefficients only odd derivatives of $f(0)$, while on the other hand B and D involve only even powers of x and in their coefficients only even derivatives of $f(0)$, we know that in the developments of sn (mx) B and D will not present themselves, nor will A and C in the developments of cn (mx) and dn (mx). We shall have therefore sn (mx) equal to each of the four quantities

$$A_1, \quad A_2, \quad C_1, \quad C_2,$$

and cn (mx) and dn (mx) to each of the four,

$$B_1, \quad B_2, \quad D_1, \quad D_2.$$

22. We shall here slightly modify the notation in our expressions for the coefficients and write

$$L_n = \frac{M_n}{n!}, \text{ and } K_n = \frac{N_n}{n!}.$$

Then

$$N_n = DM_{n-1},$$

$$M_n = [D^2 + (n-1)^2 S]\, M_{n-2} \\ + (n-1)(n-3)(n-2)^2\, TM_{n-4},$$

where

$$S = 1 \text{ and } T = 0 \text{ for } A \text{ and } B,$$

and

$$S = 1 + k^2 \text{ and } T = -k^2 \text{ for } C \text{ and } D;$$

also

$$M_0 = f(0),\ M_1 = f'(0), \text{ and } N_0 = f(0).$$

23. In order that these results may hold it is necessary first that the function we are developing shall not become infinite inside of the contour of integration; and second, that x shall satisfy the conditions of §7 or of §17.

The first requirement will be met if

$$l\,|\,m\,| < |\,K'\,|.$$

When x $\left(\text{or the ratio } \frac{x}{K}\right)$ is real this is always possible whatever the value of m, since we may take l as small as we please. If x $\left(\text{or the ratio } \frac{x}{K}\right)$ is not real, we must have, writing mx in the form $Km\xi + iK'm\eta$, the coefficient of iK' lie between -1 and $+1$.

For A and B the conditions of §7 require that

$$|\sin x\,| < 1,$$

or, if we must take $l < \log(1 + \sqrt{2})$, that

$$|\sin x\,| < \frac{e^l - e^{-l}}{2};$$

and further, that the real part of x lie between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

For C and D the conditions of §17 require, when k is real and $k^2 < 1$, that

$$|\operatorname{sn} x| < 1,$$

or, if we must take $l < l_1$, l_1 being defined as in that section, that

$$|\operatorname{sn} x| < (i \operatorname{sn} iK'l)^2,$$

and further, that the real part of x lie between $-K$ and $+K$.

In the general case, however, when k is any complex quantity, we can only say that we must have

$$|\operatorname{sn} x| < |\operatorname{sn} z|$$

on all points of the contour of integration, and that the coefficient of K, x being written in the form $K\xi + iK'\eta$, must lie between -1 and $+1$.

The Function $\varphi(x)$.

24. We shall consider first the function $\varphi(x)$ which satisfies the differential equation

$$\varphi'^2 \text{ or } \left(\frac{d\varphi}{dx}\right)^2 = m^2(P + Q\varphi^2 + R\varphi^4),$$

and therefore

$$D^2\varphi \text{ or } \varphi'' = m^2(Q\varphi + 2R\varphi^3),$$

and in our results replace P, Q and R,

$$\begin{array}{llll} \text{for } \operatorname{sn}(mx) \text{ by} & 1 & , -(1+k^2), & k^2, \\ \text{“ } \operatorname{cn}(mx) \text{ “} & 1-k^2, & 2k^2-1 & , -k^2, \\ \text{“ } \operatorname{dn}(mx) \text{ “} & k^2-1, & 2-k^2 & , -1. \end{array}$$

Since the odd and even derivatives are even and odd functions of φ of degree one higher than the order of differentiation,

$$M_{2n} = \sum_{\rho=0}^{n} M_{2n,\,2\rho+1}\,\varphi^{2\rho+1},$$

$$N_{2n+1} = \varphi' \sum_{\rho=0}^{n} (2\rho+1)\,M_{2n,\,2\rho+1}\,\varphi^{2\rho},$$

$$M_{2n+1} = \varphi' \sum_{\rho=0}^{n} (2\rho+1)\,M_{2n+1,\,2\rho+1}\,\varphi^{2\rho},$$

$$N_{2n} = \varphi'' \sum_{\rho=0}^{n-1} (2\rho+1)\,M_{2n-1,\,2\rho+1}\,\varphi^{2\rho} + \varphi'^2 \sum_{\rho=1}^{n-1} (2\rho+1)\,2\rho M_{2n-1,\,2\rho+1}\,\varphi^{2\rho-1},$$

where in φ, φ' and φ'' we make $x = 0$.

For $\quad \operatorname{sn}(mx) \quad \varphi(0) = 0,\ \varphi'(0) = m;$
for $\quad \left.\begin{array}{l}\operatorname{cn}(mx)\\ \operatorname{dn}(mx)\end{array}\right\} \begin{array}{l}\varphi(0) = 1,\ \varphi'(0) = 0,\\ \varphi''(0) = m^2(Q + 2R).\end{array}$
and

Therefore, for $\operatorname{sn}(mx)$,

$$M_{2n+1} = mM_{2n+1,1},$$
$$N_{2n+1} = mM_{2n,1},$$

and for $\operatorname{cn}(mx)$ and $\operatorname{dn}(mx)$,

$$M_{2n} = \sum_{\rho=0}^{n} M_{2n,2\rho+1},$$
$$N_{2n} = m^2(Q + 2R)\sum_{\rho=0}^{n-1}(2\rho + 1)\,M_{2n-1,2\rho+1}.$$

$M_{2n,2\rho+1}$ and $M_{2n+1,2\rho+1}$ are polynomials in k^2 and m^2 of degree n in each and have integer coefficients.

Method of Calculating the M's.

25.
$$\varphi'^2 = m^2(P + Q\varphi^2 + R\varphi^4),$$
$$\varphi'' = m^2(Q\varphi + 2R\varphi^3).$$

Now if

$$M_{n-2} = \Sigma M_{n-2,\rho}\,\varphi^\rho,$$

then

$$\begin{aligned} D^2M_{n-2} &= \varphi''\Sigma\rho M_{n-2,\rho}\,\varphi^{\rho-1} + \varphi'^2\Sigma\rho(\rho-1)\,M_{n-2,\rho}\,\varphi^{\rho-2}\\ &= m^2\Sigma M_{n-2,\rho}\{\rho(\rho-1)P\varphi^{\rho-2} + \rho^2Q\varphi^\rho + \rho(\rho+1)R\varphi^{\ +2}\}\end{aligned}$$

and

$$\begin{aligned} [D^2 + (n-1)^2S]\,M_{n-2} &= \Sigma M_{n-2,\rho}\{\rho(\rho-1)m^2P\varphi^{\rho-2}\\ &\qquad + [\rho^2m^2Q + (n-1)^2S]\,\varphi^\rho + \rho(\rho+1)\,m^2R\varphi^{\rho+2}\}\\ &= \Sigma\{(\rho-1)(\rho-2)\,m^2RM_{n-2,\rho-2}\\ &\qquad + [\rho^2m^2Q + (n-1)^2S]\,M_{n-2,\rho}\\ &\qquad + (\rho+1)(\rho+2)\,PM_{n-2,\rho+2}\}\,\varphi^\rho.\end{aligned}$$

Even Indices.

26. We may follow the method of M. André and form a table of the coefficients of the different powers of φ in the M_{2n}.

Our table will be

$$\begin{array}{llllll} M_{0,1}(=1), \\ M_{2,1} & M_{2,3}, \\ M_{4,1} & M_{4,3} & M_{4,5}, \\ \dots\dots\dots\dots\dots \\ M_{2n-2,1} & M_{2n-2,3} & M_{2n-2,5} \dots\dots & M_{2n-2,2n-1}, \\ M_{2n,1} & M_{2n,3} & M_{2n,5} \dots\dots & M_{2n,2n-1} & M_{2n,2n+1}. \end{array}$$

Each term $M_{2n,2p+1}$ of any horizontal line is formed from the nearest three terms of the line immediately above it, and from the term in the same column two rows above it, by the following operations:

We multiply

the term on the left by	$(2p-1)\,2pm^2R$,
" " directly above by	$(2p+1)^2m^2Q+(2n-1)^2S$,
" " on the right by	$(2p+2)(2p+3)\,m^2P$,
" " in the second row above by	$(2n-1)(2n-3)(2n-2)^2T$,

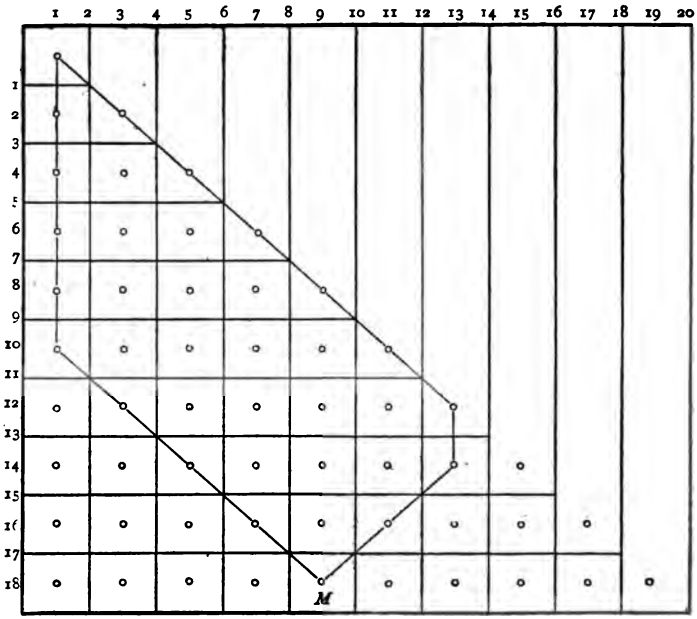
and then we add the four products so obtained.

27. We may construct a diagram representing by points the position of all the M's in the first $n+1$ rows of the table. We will place at the top of each column the odd number which is the second suffix in that column, and over the spaces between the columns the even numbers which come between these odd numbers; also, on the left against each row, the even number which is the first suffix in that row, and against the spaces between these rows the odd numbers which come between these even numbers.

28. Consider, then, the ternary paths leading from $M_{2n,2p+1}$ to $M_{0,1}$, such paths connecting $n+1$ points by n short lines or traces, which are either vertical or inclined at an angle of 45° to the right or to the left.

If we calculate that part of $M_{2n,2p+1}$ which is independent of T without making any reductions, it will consist of a series of terms, each term corresponding to one of these ternary paths and containing n factors corresponding to the n traces of that path.

Every oblique trace introduces as factors the number marked at the top of the column at which we arrive and the number marked over the space that we cross between the columns. It also introduces the factors m^2 and R or P according as we go to the left or to the right.



In every ternary path there must be at least one oblique trace inclined to the left, crossing each space between the column containing $M_{2n,\,2\rho+1}$ and the first column. These we will call essential oblique traces. Together they introduce the factor $(2\rho)!\,m^{2\rho}R^{\rho}$, which is therefore a common factor of all the terms of $M_{2n,\,2\rho+1}$. We will choose for this the lowest of the oblique traces that crosses a vertical space to the left of $M_{2n,\,2\rho+1}$.

The remaining oblique traces are in pairs, the two lines of a

pair crossing the space between the same two columns in opposite directions, the lower one towards the right and the upper one towards the left. They introduce the factor m^4PR multiplied by the product of the two numbers at the top of these columns and the square of the number between.

The factors introduced by the vertical traces are binomials of the form

$$(2\sigma + 1)^2 m^2 Q + (2\nu - 1)^2 S,$$

that is, the first term is m^2Q multiplied by the square of the number at the top of the column, and the second is S multiplied by the square of the number placed at the left against the space traversed.

29. As for the T, we notice that it takes up that portion of a ternary path which leads from one point to the second point directly above. We may consider this as made up of two vertical traces, or two oblique traces crossing the space to the left of the column containing the two points, or two oblique traces crossing the space to the right of this column. We shall adopt the last point of view, and convene, that, whenever a ternary path has one or more pairs of oblique traces, the two traces of a pair *not being separated by vertical traces, but crossing successive horizontal spaces*, such a path may be repeated and, in addition to the terms we have already considered, bring in new terms. In these terms some or all of the factors brought in by these pairs of oblique traces, instead of being each m^4PR multiplied by numbers at the top of the table, will each be T multiplied by a similar group of numerical factors found at the left of the table, namely, the odd numbers placed against the spaces crossed and the square of the even number between. This must be done in every possible way.

Odd Indices.

30. For the M's whose first suffix is odd everything is the same, but, since in the diagram (p. 39) the numbers at the left will be moved upwards one place, we shall have in the second

term of our binomial factors introduced by the vertical traces the square of an even number instead of the square of an odd number. Moreover, the groups of numerical factors brought in with the T's will consist of the product of two successive even numbers multiplied by the square of the odd number between instead of the product of two successive odd numbers multiplied by the square of the even number between, which will still be the form of the factors brought in with the (m^4PR)'s. If, then, without making any reductions, we write down the M's whose first suffix is even, we can immediately get from these the corresponding M's with odd first suffix by increasing by unity the second numbers which characterize the binomial factors introduced by the vertical traces, and each number of the groups of factors brought in with the T's.

Calculation of the M's.

31. By aid of our diagram we can write an expression for any M without having previously calculated the M's in the rows above it.

For convenience we will indicate the binomial factors (§28) by putting in a parenthesis the two numbers which characterize them. Thus,

$$(3, 7) \equiv (3^2m^2Q + 7^2S).$$

(3, 7) corresponds to the vertical trace lying in the column which has 3 written at the top and crossing the horizontal space which has 7 written against it at the left.

When the numbers in such a parenthesis have a common factor, that factor may be taken out and its square written as a numerical coefficient, thus:

$$(3, 3) = 3^2(1, 1).$$

Even Indices.

32. These M's will take the following form, which we can write at once from the diagram by following the principles laid down above.

$$M_{0,1} = 1;$$
$$M_{2,1} = (1, 1),$$
$$M_{2,3} = 2m^2R;$$
$$M_{4,1} = (1, 1)(1, 3) + 1.3.2^2(m^4PR + T),$$
$$M_{4,3} = 2m^2R[(1, 1) + (3, 3)] = 20m^2R(1, 1),$$
$$M_{4,5} = 4!\, m^4R^2;$$
$$\begin{aligned} M_{6,1} &= (1, 1)(1, 3)(1, 5) + 1.3.2^2m^4PR[(1, 1) + (3, 3) + (1, 5)] \\ &\qquad + T[1.3.2^2(1, 5) + 3.5.4^2(1, 1)] \\ &= (1, 1)(1, 3)(1, 5) + 12m^4PR[10(1, 1) + (1, 5)] \\ &\qquad + 12T[20(1, 1) + (1, 5)], \end{aligned}$$
$$\begin{aligned} M_{6,3} &= 2m^2R\,\{(1, 1)(1, 3) + (1, 1)(3, 5) + (3, 3)(3, 5) \\ &\qquad + (m^4PR + T)(1.3.2^2 + 3.5.4^2)\}. \\ &= 2m^2R\{(1, 1)[(1, 3) + 10(3, 5)] + 252(m^4PR + T)\}, \end{aligned}$$
$$M_{6,5} = 4!\, m^4R^2[(1, 1) + (3, 3) + (5, 5)] = 35.4!\, m^4R^2(1, 1),$$
$$M_{6,7} = 6!\, m^6R^3;$$
etc. = etc.

Odd Indices.

33. All we have to do is to take the forms written above before reduction and increase by unity certain of the numbers as directed above (§30).

$$M_{1,1} = 1;$$
$$M_{3,1} = (1, 2),$$
$$M_{3,3} = 2m^2R;$$
$$M_{5,1} = (1, 2)(1, 4) + 1.3.2^2m^4PR + 2.4.3^2T,$$
$$M_{5,3} = 2m^2R[(1, 2) + (3, 4)],$$
$$M_{5,5} = 4!\, m^4R^2;$$
$$\begin{aligned} M_{7,1} &= (1, 2)(1, 4)(1, 6) + 1.3.2^2m^4PR[(1, 2) + (3, 4) + (1, 6)] \\ &\qquad + T[2.4.3^2(1, 6) + 4.6.5^2(1, 2)], \end{aligned}$$
$$\begin{aligned} M_{7,3} &= 2m^2R\,\{(1, 2)(1, 4) + (1, 2)(3, 6) + (3, 4)(3, 6) \\ &\qquad + m^4PR(1.3.2^2 + 3.5.4^2) + T(2.4.3^2 + 4.6.5^2)\} \\ &= 2m^2R\{(1, 2)[(1, 4) + 9(1, 2) + 9(3, 4)] \\ &\qquad + 252m^4PR + 672T\}, \end{aligned}$$
$$M_{7,5} = 4!\, m^4R^2[(1, 2) + (3, 4) + (5, 6)],$$
$$M_{7,7} = 6!\, m^6R^3;$$
etc. = etc.

Resulting Series.

34. In these expressions for the M's are now to be substituted the values of P, Q, R, S and T, as given in §§22 and 24. Then the formulae of §24 will give us the coefficients of our developments.

I.—sn (mx).

35. Here $PR = k^2$ and $Q = -(1 + k^2)$.
For A_1 and A_2 $S = 1$ and $T = 0$. Thus we have

$$\begin{aligned}
M_{0,1} &= 1, \\
M_{2,1} &= -(m^2k^2 + m^2 - 1), \\
M_{4,1} &= (m^2k^2 + m^2 - 1)(m^2k^2 + m^2 - 3^2) + 12m^4k^2 \\
&= m^4k^4 + 2(7m^2 - 5)m^2k^2 + (m^2 - 1)(m^2 - 3^2),
\end{aligned}$$

etc.

These values multiplied by m give the coefficients in the development A_1 for sn (mx). Similarly we calculate A_2 for sn (mx).

For C_1 and C_2

$$S = -Q = 1 + k^2,$$
$$PR = -T = k^2.$$

In these two cases $1 + k^2$ is a factor of both terms of our binomial expressions; thus the k^2's and m^2's separate of themselves and the M's appear arranged in powers of $1 + k^2$ and k^2 without any reduction. Moreover, they are neater in this form than if arranged according to powers of k^2 alone.

For example,

$$\begin{aligned}
M_{6,1} = &-(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)(1 + k^2)^3 \\
&- 12m^4[10(m^2 - 1^2) + (m^2 - 5^2)](1 + k^2)k^2 \\
&+ 12[20(m^2 - 1^2) + (m^2 - 5^2)](1 + k^2)k^2 \\
= &-(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)(1 + k^2)^3 \\
&- 12(m^2 - 1)(11m^4 - 24m^2 - 45)(1 + k^2)k^2.
\end{aligned}$$

II.—cn (mx).

36. Here

$$P = 1 - k^2, \quad Q = 2k^2 - 1, \quad R = -k^2.$$

For B_2 and D_2 we can get a formula easier to use than the formula given in §24.

The equation of §25 gives us, when we make $x = 0$ (except for the part involving T),

$$\Sigma M_{n,\rho} = \Sigma M_{n-2,\rho}\} \rho(\rho - 1) m^2 P + [\rho^2 m^2 Q + (n-1)^2 S] + \rho(\rho+1) m^2 R \}.$$

But since

$$P + Q + R = 0,$$

this reduces to

$$\begin{aligned} \Sigma M_{n,\rho} &= \Sigma M_{n-2,\rho}\} \rho m^2 (R - P) + (n-1)^2 S \} \\ &= m^2 (R - P)\, \Sigma \rho M_{n-2,\rho} + (n-1)^2 S \Sigma M_{n-2,\rho}; \end{aligned}$$

or, taking account also of T and putting $R - P = -1$,

$$\Sigma M_{n,\rho} = -m^2 \Sigma \rho M_{n-2,\rho} + (n-1)^2 S \Sigma M_{n-2,\rho} + (n-1)(n-3)(n-2)^2 T \Sigma M_{n-4,\rho}.$$

For example, take D_2. Then

$$S = 1 + k^2, \quad T = -k^2;$$

and suppose we have already found

$$\begin{aligned} M_{0.1} &= 1, \\ M_{2,1} &= (2m^2 + 1) k^2 - (m^2 - 1), \\ M_{2,3} &= -2m^2 k^2, \end{aligned}$$

and therefore

$$M_2 = k^2 - (m^2 - 1).$$

Then the above formula gives for M_4,

$$\begin{aligned} M_4 &= -m^2 \left\{ \begin{matrix} (2m^2 + 1) k^2 - (m^2 - 1) \\ -6m^2 k^2 \end{matrix} \right\} \\ &\qquad + 3^2 (1 + k^2) \} k^2 - (m^2 - 1) \{ - 1.3.2^2 k^2 \\ &= 3^2 k^4 + 2(m^2 - 1)(2m^2 - 3) k^2 + (m^2 - 1)(m^2 - 3^2). \end{aligned}$$

The work would have been greater if we had written down the values of $M_{4,1}$, $M_{4,3}$, $M_{4,5}$ and added them.

III.—dn (mx).

37. We can dispense with the separate calculation of these developments and write them directly from the forms we have already obtained for cn (mx).

The values of

	P,	Q,	R
for cn (mx) are	$1-k^2$,	$2k^2-1$,	$-k^2$;
for dn (mx) are	k^2-1,	$2-k^2$,	-1.

If instead of these we write

$$k_2^2-k_1^2,\quad 2k_1^2-k_2^2,\quad -k_1^2,$$

we shall make P, Q and R homogeneous and get results that will give the formulae

for cn (mx) if we make $k_1=k$, $k_2=1$;
and for dn (mx) " " $k_1=1$, $k_2=k$.

38. For D_1 and D_2 we may also make

$$S=k_1^2+k_2^2 \text{ and } T=-k_1^2k_2^2,$$

and our M's and N's will be homogeneous expressions in k_1 and k_2,

$$M_{2n,\,2p+1} \text{ and } M_{2n+1,\,2p+1} \text{ of degree } 2n,$$

and therefore

$$M_{2n} \text{ and } N_{2n} \text{ of degree } 2n.$$

If, then, we have calculated D_1 and D_2 for cn (mx) and wish to do the same for dn (mx), it is only necessary, by the introduction of a new k, to make each term a homogeneous expression of degree equal to the degree of the sn which it multiplies, and then remove all the old k's. But that is the same as changing the exponent of each k into the number which would be required to raise it to the same degree as the sn which it multiplies (supposing we have introduced the factor k^0 wherever k is absent).

For example, the third term of the development which D_1 gives for cn (mx) is

$$\frac{m^2}{4!}\Big\{4(m^2-1)k^2+(m^2-2^2)\Big\}\,\mathrm{sn}^4x.$$

therefore the corresponding term in the same development for dn (mx) will be

$$\frac{m^2k^2}{4!}\Big\{(m^2-2^2)k^2+4(m^2-1)\Big\}\,\mathrm{sn}^4x.$$

39. For B_1 and B_2

$$T = 0 \text{ and } S = 1,$$

and the M's will no longer be homogeneous. But if we recall their mode of formation, we may remember that both by the vertical and by the oblique traces m^2 and P, Q or R are always brought in together, while there is no m^2 brought in with S or T. Therefore, from the developments B_1 and B_2 for $\operatorname{cn}(mx)$, those for $\operatorname{dn}(mx)$ may be obtained at once by changing the exponent of each k into the number which would be necessary to raise it to the same degree as the m which it multiplies.

For example, the third term which B_2 gives for $\operatorname{cn}(mx)$ is $\cos x$ multiplied by

$$\frac{1}{4!}\left\{4m^4k^2 + (m^2 - 1^2)(m^2 - 3^2)\right\}\sin^4 x,$$

therefore the corresponding term in the same development for $\operatorname{dn}(mx)$ is $\cos x$ multiplied by

$$\frac{1}{4!}\left\{4m^4k^2 + (m^2k^2 - 1^2)(m^2k^2 - 3^2)\right\}\sin^4 x,$$

or, arranged according to powers of k^2,

$$\frac{1}{4!}\left\{m^4k^4 + 2(2m^2 - 5)m^2k^2 + 1^2.3^2\right\}\sin^4 x.$$

This relationship between $\operatorname{cn}(mx)$ and $\operatorname{dn}(mx)$ is reciprocal, and either set of expressions may be obtained from the other by the rules given above.

TABLE.

$$(1)\quad \operatorname{sn}(mx) = m\sin x - \frac{m}{3!}\left\{m^2k^2 + (m^2 - 1)\right\}\sin^3 x$$
$$+ \frac{m}{5!}\left\{\begin{array}{l} m^4k^4 + 2(7m^2 - 5)m^2k^2 \\ \quad + (m^2 - 1^2)(m^2 - 3^2) \end{array}\right\}\sin^5 x$$
$$- \frac{m}{7!}\left\{\begin{array}{l} m^6k^6 + 5(27m^2 - 7)m^4k^4 \\ \quad + (135m^4 - 490m^2 + 259)m^2k^2 \\ \quad + (m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2) \end{array}\right\}\sin^7 x$$
$$+ \text{etc.}$$

(2) $$\operatorname{sn}(mx) = \cos x \Bigg[m \sin x - \frac{m}{3!}\left\{m^2k^2 + (m^2 - 2^2)\right\}\sin^3 x$$
$$+ \frac{m}{5!}\left\{\begin{array}{l} m^4k^4 + 2(7m^2 - 10)m^2k^2 \\ + (m^2 - 2^2)(m^2 - 4^2) \end{array}\right\}\sin^5 x$$
$$- \frac{m}{7!}\left\{\begin{array}{l} m^6k^6 + (135m^2 - 56)m^4k^4 \\ + (135m^4 - 784m^2 + 784)m^2k^2 \\ + (m^2 - 2^2)(m^2 - 4^2)(m^2 - 6^2) \end{array}\right\}\sin^7 x$$
$$+ \text{etc.}\Bigg].$$

(3) $$\operatorname{sn}(mx) = m \operatorname{sn} x - \frac{m}{3!}(m^2 - 1^2)(1 + k^2)\operatorname{sn}^3 x$$
$$+ \frac{m}{5!}\left\{\begin{array}{l} (m^2 - 1^2)(m^2 - 3^2)(1 + k^2)^2 \\ + 12(m^4 - 1)k^2 \end{array}\right\}\operatorname{sn}^5 x$$
$$- \frac{m}{7!}\left\{\begin{array}{l} (m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)(1 + k^2)^3 \\ + 12(m^2 - 1)(11m^4 - 24m^2 - 45)(1 + k^2)k^2 \end{array}\right\}\operatorname{sn}^7 x$$
$$+ \text{etc.}$$

(4) $$\operatorname{sn}(mx) = \operatorname{cn} x \operatorname{dn} x \Bigg[m \operatorname{sn} x - \frac{m}{3!}(m^2 - 2^2)(1 + k^2)\operatorname{sn}^3 x$$
$$+ \frac{m}{5!}\left\{\begin{array}{l} (m^2 - 2^2)(m^2 - 4^2)(1 + k^2)^2 \\ + 12(m^4 - 6)k^2 \end{array}\right\}\operatorname{sn}^5 x$$
$$- \frac{m}{7!}\left\{\begin{array}{l} (m^2 - 2^2)(m^2 - 4^2)(m^2 - 6^2)(1 + k^2)^3 \\ + 12(m^2 - 2^2)(11m^4 - 12m^2 - 104)(1 + k^2)k^2 \end{array}\right\}\operatorname{sn}^7 x$$
$$+ \text{etc.}\Bigg].$$

(5) $$\operatorname{cn}(mx) = 1 - \frac{m^2}{2}\sin^2 x$$
$$+ \frac{m^2}{4!}\left\{4m^2k^2 + (m^2 - 2^2)\right\}\sin^4 x$$
$$- \frac{m^2}{6!}\left\{\begin{array}{l} 16m^4k^4 + 4(11m^2 - 20)m^2k^2 \\ + (m^2 - 2^2)(m^2 - 4^2) \end{array}\right\}\sin^6 x$$
$$+ \text{etc.}$$

(6) $$\operatorname{cn}(mx) = \cos x \Bigg[1 - \frac{1}{2}(m^2 - 1^2)\sin^2 x$$
$$+ \frac{1}{4!}\left\{4m^4k^2 + (m^2 - 1^2)(m^2 - 3^2)\right\}\sin^4 x$$
$$- \frac{1}{6!}\left\{\begin{array}{l} 16m^6k^4 + 4(11m^2 - 35)m^4k^2 \\ + (m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2) \end{array}\right\}\sin^6 x$$
$$+ \text{etc.}\Bigg].$$

$$
\begin{aligned}
(7)\ \operatorname{cn}(mx) = {} & 1 - \frac{m^2}{2}\operatorname{sn}^2 x \\
& + \frac{m^2}{4!}\Big\{4(m^2-1)k^2 + (m^2-2^2)\Big\}\operatorname{sn}^4 x \\
& - \frac{m^2}{6!}\left\{\begin{array}{l} 16(m^2-1)(m^2-4)k^4 \\ \quad + 4(m^2-1)(11m^2-14)k^2 \\ \quad + (m^2-2^2)(m^2-4^2) \end{array}\right\}\operatorname{sn}^6 x \\
& + \text{etc.}
\end{aligned}
$$

$$
\begin{aligned}
(8)\ \operatorname{cn}(mx) = {} & \operatorname{cn} x \operatorname{dn} x\Bigg[1 + \frac{1}{2}\Big\{k^2 - (m^2-1^2)\Big\}\operatorname{sn}^2 x \\
& + \frac{1}{4!}\left\{\begin{array}{l} 1^2.3^2k^4 + 2(m^2-1)(2m^2-3)k^2 \\ \quad + (m^2-1^2)(m^2-3^2) \end{array}\right\}\operatorname{sn}^4 x \\
& + \frac{1}{6!}\left\{\begin{array}{l} 1^2.3^2.5^2k^6 - (m^2-1)(16m^4-124m^2+135)k^4 \\ \quad - (m^2-1)(44m^4-131m^2+135)k^2 \\ \quad - (m^2-1^2)(m^2-3^2)(m^2-5^2) \end{array}\right\}\operatorname{sn} x \\
& + \text{etc.}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
(9)\ \operatorname{dn}(mx) = {} & 1 - \frac{m^2k^2}{2}\sin^2 x \\
& + \frac{m^2k^2}{4!}\Big\{m^2k^2 + 4(m^2-1)\Big\}\sin^4 x \\
& - \frac{m^2k^2}{6!}\left\{\begin{array}{l} m^4k^4 + 4(11m^2-5)m^2k^2 \\ \quad + 16(m^2-1)(m^2-4) \end{array}\right\}\sin^6 x \\
& + \text{etc.}
\end{aligned}
$$

$$
\begin{aligned}
(10)\ \operatorname{dn}(mx) = {} & \cos x\Bigg[1 - \frac{1}{2}(m^2k^2-1)\sin^2 x \\
& + \frac{1}{4!}\Big\{m^4k^4 + 2(2m^2-5)m^2k^2 + 1^2.3^2\Big\}\sin^4 x \\
& - \frac{1}{6!}\left\{\begin{array}{l} m^6k^6 + (44m^2-35)m^4k^4 \\ \quad + (16m^4-140m^2+259)m^2k^2 - 1^2.3^2.5^2 \end{array}\right\}\sin^6 x \\
& + \text{etc.}\Bigg].
\end{aligned}
$$

$$
\begin{aligned}
(11)\ \operatorname{dn}(mx) = {} & 1 - \frac{m^2k^2}{2}\operatorname{sn}^2 x \\
& + \frac{m^2k^2}{4!}\Big\{(m^2-2^2)k^2 + 4(m^2-1)\Big\}\operatorname{sn}^4 x \\
& - \frac{m^2k^2}{6!}\left\{\begin{array}{l} (m^2-2^2)(m^2-4^2)k^4 \\ \quad + 4(m^2-1)(11m^2-14)k^2 \\ \quad + 16(m^2-1)(m^2-4) \end{array}\right\}\operatorname{sn}^6 x \\
& + \text{etc.}
\end{aligned}
$$

$$(12)\quad \operatorname{dn}(mx) = \operatorname{cn} x \operatorname{dn} x \Bigg[1 - \frac{1}{2}\Big\{(m^2 - 1^2)k^2 - 1^2\Big\}\operatorname{sn}^2 x$$

$$+ \frac{1}{4!}\left\{\begin{matrix}(m^2 - 1^2)(m^2 - 3^2)k^4 \\ \quad + 2(m^2 - 1)(2m^2 - 3)k^2 + 1^2.3^2\end{matrix}\right\}\operatorname{sn}^4 x$$

$$- \frac{1}{6!}\left\{\begin{matrix}(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)k^6 \\ \quad + (m^2 - 1)(44m^4 - 131m^2 + 135)k^4 \\ \quad + (m^2 - 1)(16m^4 - 124m^2 + 135)k^2 \\ \quad - 1^2.3^2.5^2\end{matrix}\right\}\operatorname{sn}^6 x$$

$$+ \text{etc.}\Bigg].$$